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**LETTER TO THE EDITOR**

# Darboux transformations and linear parabolic partial differential equations

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**Abstract**
Solutions for a class of linear parabolic partial differential equation are provided. These solutions are obtained by first solving a system of $(n + 1)$ nonlinear partial differential equations. This system arises as the coefficients of a Darboux transformation and is equivalent to a matrix Burgers' equation. This matrix equation is solved using a generalized Hopf–Cole transformation. The solutions for the original equation are given in terms of solutions of the heat equation. These results are applied to the $(1 + 1)$-dimensional Schrödinger equation where all bound state solutions are obtained for a $2n$-parameter family of potentials. As a special case, the solutions for integral members of the regular and modified Pöschl–Teller potentials are recovered.

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## 1. Introduction

The study of linear partial differential equations (PDEs) has a long well-established history. In addition to its mathematical elegance, the field of linear PDEs has many important applications including heat transfer, vibrations in elastic solids and wave propagation. The standard technique for obtaining exact solutions is the separation of variables. This requires the integration of a system of ordinary differential equations and leads to solutions in an eigenfunction expansion. An alternative technique is the use of Darboux transformations [3], where solutions of different differential equations are linked. An example is given by Englefield [4] who showed that solutions of the $(1 + 1)$-dimensional Schrödinger equation (with $\hbar = 2m = 1$):

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} - 2\beta^2 \mathrm{sech}^2(\beta x)\psi \tag{1}$$

can be generated from solutions of the free particle equation,

$$\mathrm{i}\frac{\partial \tilde{\psi}}{\partial t} = -\frac{\partial^2 \tilde{\psi}}{\partial x^2} \tag{2}$$

via the Darboux transformation:

$$\psi = \frac{\partial \tilde{\psi}}{\partial x} - \beta \tanh(\beta x)\tilde{\psi}. \tag{3}$$

More recently, there has been a renewed interest in Darboux transformations and their applications to Schrödinger's equation. In a two part paper, Bagrov and Samsonov [1] and Samsonov and Ovcharov [11] considered the time-independent Schrödinger equation. Using a composition of $n$ first-order Darboux transformation, they determine new classes of solvable Schrödinger equations. These new classes of solvable equations have potentials that are related to the harmonic oscillator, Morse, and effective Coulomb potentials. An extension to the time-dependent Schrödinger equation has also been given by Bagrov *et al* [2] but requires an additional assumption to determine the second-order Darboux transformation. The survey paper by Rosu [9] traces many of the crucial steps in the development of Darboux transformations. In [7] Matveev describes the solutions associated with a composition of $n$ first-order Darboux transformations for a general class of linear evolution equations, though the primary focus of his work is on nonlinear evolution equations.

In this letter, a generalized $n+1$st-order Darboux transformation is constructed that relates the solutions of the heat equation with a nonzero potential to the heat equation with a zero potential. It is shown that this Darboux transformation can be found explicitly without the use of additional assumptions. These results are applied to the (1 + 1) Schrödinger equation where solutions are obtained for a new class of multi-well potentials.

The letter is organized as follows. In section 2, a system of $(n + 1)$ nonlinear partial differential equations is derived for the coefficients of the Darboux transformation. Rewriting this system as a matrix partial differential equation, a generalized Hopf–Cole transformation is introduced that linearizes the matrix equation and allows for their solution. In section 3, these results are applied to the (1 + 1) Schrödinger equation where it is shown that separable solutions of the free particle Schrödinger equation lead to solutions of Schrödinger equation with multi-well potentials. In section 4, the bound states are obtained. Finally, in section 5, the integral members of the regular and modified Pöschl–Teller potentials are recovered as special cases.

## 2. Darboux transformations

In what follows all functions are assumed to be $C^\infty(\mathbb{R})$. Consider the heat equation with a nonzero potential

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t, x)\, u. \tag{4}$$

Introducing the Darboux transformation

$$u = \frac{\partial^{n+1} v}{\partial x^{n+1}} + \sum_{i=0}^{n} A_i(t, x)\, \frac{\partial^i v}{\partial x^i} \tag{5}$$

followed by the substitution of (5) into (4) leads to

$$\frac{\partial^{n+2} v}{\partial x^{n+1}\partial t} + \sum_{i=0}^{n}\left(A_i \frac{\partial^{i+1} v}{\partial x^i \partial t} + \frac{\partial A_i}{\partial t}\frac{\partial^i v}{\partial x^i}\right) = \frac{\partial^{n+3} v}{\partial x^{n+3}} + A_i \frac{\partial^{i+2} v}{\partial x^{i+2}} + 2\frac{\partial A_i}{\partial x}\frac{\partial^{i+1} v}{\partial x^{i+1}}$$
$$+ \frac{\partial^2 A_i}{\partial x^2}\frac{\partial^i v}{\partial x^i} + f\left(\frac{\partial^{n+1} v}{\partial x^{n+1}} + \sum_{i=0}^{n} A_i \frac{\partial^i v}{\partial x^i}\right). \tag{6}$$

Requiring that $v$ satisfy the heat equation and rearranging the terms give

$$\left(-f-2\frac{\partial A_n}{\partial x}\right)\frac{\partial^{n+1}v}{\partial x^{n+1}}+\sum_{i=1}^{n}\left(\frac{\partial A_i}{\partial t}-\frac{\partial^2 A_i}{\partial x^2}-2\frac{\partial A_{i-1}}{\partial x}-fA_i\right)\frac{\partial^i v}{\partial x^i}$$
$$+\left(\frac{\partial A_0}{\partial t}-\frac{\partial^2 A_0}{\partial x^2}-fA_0\right)v=0. \tag{7}$$

Since the solutions of the heat equation that appear in equation (7) can be varied arbitrarily, this requires that the coefficients of equation (7) involving $v$ and its higher order derivatives must vanish in order for (7) to be satisfied. This leads to the following system of partial differential equations for the functions $A_i$ and $f$:

$$\frac{\partial A_0}{\partial t}-\frac{\partial^2 A_0}{\partial x^2}-fA_0=0 \tag{8a}$$

$$\frac{\partial A_i}{\partial t}-\frac{\partial^2 A_i}{\partial x^2}-2\frac{\partial A_{i-1}}{\partial x}-fA_i=0 \qquad \text{for} \quad i=1,2,\ldots,n \tag{8b}$$

$$f+2\frac{\partial A_n}{\partial x}=0. \tag{8c}$$

Further, eliminating $f$ from (8$a$) and (8$b$) using (8$c$) yields

$$\frac{\partial A_0}{\partial t}-\frac{\partial^2 A_0}{\partial x^2}+2A_0\frac{\partial A_n}{\partial x}=0 \tag{9a}$$

$$\frac{\partial A_i}{\partial t}-\frac{\partial^2 A_i}{\partial x^2}-2\frac{\partial A_{i-1}}{\partial x}+2A_i\frac{\partial A_n}{\partial x}=0 \qquad \text{for} \quad i=1,2,\ldots,n. \tag{9b}$$

This system of PDEs can conveniently be written in the matrix Burger's form

$$\mathbf{\Omega}_t+2\mathbf{\Omega}_x\mathbf{\Omega}-\mathbf{\Omega}_{xx}=0 \tag{10}$$

where $\mathbf{\Omega}$ is the $(n+1)\times(n+1)$ matrix given by

$$\mathbf{\Omega}=\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 \\ A_0 & A_1 & \cdots & A_{n-1} & A_n \end{bmatrix}. \tag{11}$$

Following Levi *et al* [6], we introduce the matrix Hopf–Cole transformation

$$\mathbf{\Omega}=-\mathbf{\Phi}_x\mathbf{\Phi}^{-1} \tag{12}$$

where substitution of (12) into equation (10) leads to the linear heat matrix equation

$$\mathbf{\Phi}_t=\mathbf{\Phi}_{xx}. \tag{13}$$

Thus, the entries $\phi_{ij}$ of the matrix $\mathbf{\Phi}$ are solutions of the heat equation. Rearranging (12) as

$$\mathbf{\Omega\Phi}=-\mathbf{\Phi}_x. \tag{12$'$}$$

and comparing the entries of the matrices in (12$'$) give

$$\phi_{i+1,j}=\frac{\partial\phi_{i,j}}{\partial x} \tag{14a}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n+1$, and

$$\sum_{i=1}^{n+1} A_{i-1}\phi_{i,j} = -\frac{\partial \phi_{n+1,j}}{\partial x} \tag{14b}$$

for $j = 1, 2, \ldots, n+1$. Renaming

$$\phi_{1,j} = \omega_j \qquad \text{for} \quad j = 1, 2, \ldots, n+1 \tag{15}$$

gives (14*a*) as

$$\phi_{i,j} = \frac{\partial^{i-1}\omega_j}{\partial_x^{i-1}} \tag{14a$'$}$$

for $i = 1, 2, \ldots, n+1$ and $j = 1, 2, \ldots, n+1$, and this, in turn, gives (14*b*) as

$$\frac{\partial^{n+1}\omega_j}{\partial x^{n+1}} + \sum_{i=0}^{n} A_i \frac{\partial^i \omega_j}{\partial x^i} = 0 \tag{14b$'$}$$

for $j = 1, 2, \ldots, n+1$. Solving the system of equations (14*b*′) for the $A_i$ gives

$$A_i = (-1)^{n-i+1}\frac{W_{\widehat{i}}}{W} \tag{16}$$

where $W_{\widehat{i}}$ and $W$ are given by the determinants

$$W_{\widehat{i}} = \begin{vmatrix} \omega_1 & \omega_2 & \cdots & \omega_n & \omega_{n+1} \\ \partial_x\omega_1 & \partial_x\omega_2 & \cdots & \partial_x\omega_n & \partial_x\omega_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \partial_x^{i-1}\omega_1 & \partial_x^{i-1}\omega_2 & \cdots & \partial_x^{i-1}\omega_n & \partial_x^{i-1}\omega_{n+1} \\ \partial_x^{i+1}\omega_1 & \partial_x^{i+1}\omega_2 & \cdots & \partial_x^{i+1}\omega_n & \partial_x^{i+1}\omega_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \partial_x^{n+1}\omega_1 & \partial_x^{n+1}\omega_2 & \cdots & \partial_x^{n+1}\omega_n & \partial_x^{n+1}\omega_{n+1} \end{vmatrix} \tag{17}$$

and

$$W = \begin{vmatrix} \omega_1 & \cdots & \omega_{n+1} \\ \vdots & \ddots & \vdots \\ \partial_x^n\omega_1 & \cdots & \partial_x^n\omega_{n+1} \end{vmatrix}. \tag{18}$$

Thus, the solutions of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t, x)u \tag{4}$$

are given by

$$u = \frac{\partial^{n+1}\nu}{\partial x^{n+1}} + \sum_{i=0}^{n} (-1)^{n-i+1}\frac{W_{\widehat{i}}}{W}\frac{\partial^i \nu}{\partial x^i} \tag{19}$$

where

$$f = 2\frac{\partial^2}{\partial x^2}(\ln W) \tag{20}$$

and the $W_{\widehat{i}}$ are given in (17) while $\nu$ is any solution of the heat equation. Hereafter, the solutions $\omega_i$ found in $W$ will be called the ‘seed’ solutions, as they are used to define the source term $f$ while $\nu$ will be called a 'generating' solution, since varying this solution to the heat equation generates different solutions to (4). Note that solution (19) can also be conveniently written as

$$u = \frac{W_\nu}{W} \tag{21}$$

where $W_\nu$ is

$$W_\nu = \begin{vmatrix} \omega_1 & \cdots & \omega_{n+1} & \nu \\ \vdots & \ddots & \vdots & \vdots \\ \partial_x^n \omega_1 & \cdots & \partial_x^n \omega_{n+1} & \partial_x^n \nu \\ \partial_x^{n+1} \omega_1 & \cdots & \partial_x^{n+1} \omega_{n+1} & \partial_x^{n+1} \nu \end{vmatrix}. \tag{22}$$

## 3. Schrödinger's equation with multi-well potentials

In this section, Schrödinger's equation with a variety of multi-well potentials is examined. The substitutions

$$t \to \mathrm{i}t \qquad x \to x \qquad \text{and} \qquad V(t,x) = -f(\mathrm{i}t, x) \tag{23}$$

transform equation (4) into Schrödinger's equation

$$\mathrm{i}\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + V(t,x)u. \tag{24}$$

Using the solutions

$$\omega_1 = \mathrm{e}^{\mathrm{i}a_1^2 t}\cosh(a_1 x + b_1) \qquad \text{and} \qquad \omega_2 = \mathrm{e}^{\mathrm{i}a_2^2 t}\sinh(a_2 x + b_2) \tag{25}$$

to Schrödinger's free particle equation, where $a_i$ and $b_i$ are arbitrary constants, gives $W$ in (18) as

$$\begin{aligned} W = {} & \mathrm{e}^{\mathrm{i}(a_1^2+a_2^2)t}(a_2 - a_1)\cosh((a_2 + a_1)x + (b_2 + b_1)) \\ & + \mathrm{e}^{\mathrm{i}(a_1^2+a_2^2)t}(a_2 + a_1)\cosh((a_2 - a_1)x + (b_2 - b_1)) \end{aligned} \tag{26}$$

which, in turn, gives $V = -f$ in (20) as

$$\frac{-2\left(a_2^2 - a_1^2\right)\left(a_2^2\cosh^2(a_1 x + b_1) + a_1^2\sinh^2(a_2 x + b_2)\right)}{W^2}. \tag{27}$$

Figure 1 shows examples of potentials when $a_1 = 1$, $b_1 = 0$, $b_2 = 0$ and $a_2$ is varied from 1.25 to 2. The most pronounced double well occurs when $a_2 = 1.25$ and converts into a single well as $a_2$ increases in value.

Similarly, if the following three seed solutions:

$$\omega_1 = \mathrm{e}^{\mathrm{i}a_1^2 t}\cosh(a_1 x + b_1) \qquad \omega_2 = \mathrm{e}^{\mathrm{i}a_2^2 t}\sinh(a_2 x + b_2) \qquad \omega_3 = \mathrm{e}^{\mathrm{i}a_3^2 t}\cosh(a_3 x + b_3) \tag{28}$$

to Schrödinger's free particle equation are chosen, more complicated potentials are obtained. This potential is illustrated in figure 2. Here $b_i = 0$, $a_1 = 1$, $a_2 = 1.5$ and $a_3$ varies from 1.6 to 2.2. When $a_3 = 1.6$ there are three wells. As $a_3$ increases in value to 2.2, the three wells convert to a single well. Continuing in this fashion, if the following $n$ seed solutions of the Schrödinger's free particle equation are chosen as

$$\omega_k = \begin{cases} \mathrm{e}^{\mathrm{i}(a_k)^2 t}\cosh(a_k x + b_k) & \text{if } k \text{ is odd} \\ \mathrm{e}^{\mathrm{i}(a_k)^2 t}\sinh(a_k x + b_k) & \text{if } k \text{ is even} \end{cases} \tag{29}$$

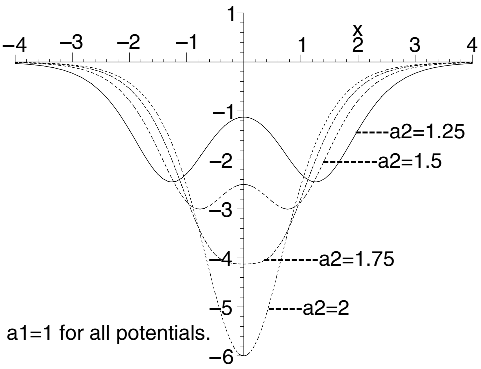


**Figure 1.** The family of potentials using two seeds.

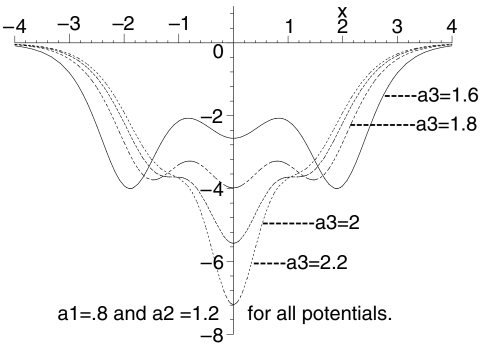


**Figure 2.** The family of potentials using three seeds.

where $a_k$ are constants, $k = 1, 2, \ldots, n$ and $a_k < a_{k+1}$, gives potentials that have between 1 and $n$ wells. The potential obtained from (20) when using the $n$ seeds given in (29) is

$$V = -2\frac{\partial^2}{\partial x^2}(\ln W) \tag{30}$$

where

$$W = \frac{1}{2^{n-1}} \sum_{\tau \in T_n} \left\{(-1)^S k_\tau \cosh(\alpha_\tau x + \beta_\tau)\right\}. \tag{31}$$

Here

$$T_n = \{\tau \mid \tau : \{1, 2, \ldots, n-1\} \to \{-1, 1\}\} \tag{32a}$$

$$S = \sum_{s=1}^{[\frac{n}{2}]} \frac{1 - \tau(2s)}{2} \tag{32b}$$

$$k_\tau = \prod_{i=1}^{n-1} (a_n - \tau(i)a_i) \prod_{k>j}^{n-1} (\tau(k)a_k - \tau(j)a_j) \tag{32c}$$

$$\alpha_\tau = \left| a_n + \sum_{l=1}^{n-1} \tau\,(l)\,a_l \right| \tag{32d}$$

and

$$\beta_\tau = \left| b_n + \sum_{l=1}^{n-1} \tau(l) b_l \right| . \tag{32e}$$

If $c_\tau = (-1)^S k_\tau$, then $V$ simplifies to

$$V = \frac{-2\sum_{\sigma,\tau\in T_n} c_\tau c_\sigma \alpha_\tau \alpha_\sigma \cosh((\alpha_\tau - \alpha_\sigma)x + (\beta_\tau - \beta_\sigma))}{\left(\sum_{\rho\in T_n} c_\rho \cosh(\alpha_\rho x + \beta_\rho)\right)^2}. \tag{33}$$

Close observation of the terms $c_\tau c_\sigma$ appearing in equation (33) shows that $V < 0$. Note: if all $b_k = 0$, the potential $V$ will be symmetric about $x = 0$, due to the symmetry properties of the hyperbolic cosine.

## 4. Bound state solutions

In this section, all the normalized bound state solutions for the class of multi-well potentials described in section 3 are given.

The general two-well potential (27) was obtained using seed solutions given in (25). To obtain the two bound state solutions for this potential, it is sufficient to use the following two generating solutions:

$$v_1 = \mathrm{e}^{\mathrm{i}a_1^2 t} \sinh(a_1 x + b_1) \qquad \text{and} \qquad v_2 = \mathrm{e}^{\mathrm{i}a_2^2 t} \cosh(a_2 x + b_2). \tag{34}$$

Substitution of (34) into (21) gives rise to the following normalized bound state solutions:

$$\psi_1 = \frac{\mathrm{e}^{\mathrm{i}a_1^2 t} \sinh(a_1 x + b_1)}{\sqrt{2a_1\left(a_2^2 - a_1^2\right)} \cdot W} \tag{35a}$$

$$\psi_2 = \frac{\mathrm{e}^{\mathrm{i}a_2^2 t} \cosh(a_2 x + b_2)}{\sqrt{2a_2\left(a_2^2 - a_1^2\right)} \cdot W} \tag{35b}$$

where $W$ is given in (26). Their energy levels are given respectively by

$$E_1 = -a_1^2 \qquad \text{and} \qquad E_2 = -a_2^2. \tag{36}$$

In the case of the three-well potential, the following generating solutions are chosen:

$$\begin{aligned} v_1 &= \mathrm{e}^{\mathrm{i}a_1^2 t} \sinh(a_1 x + b_1) \\ v_2 &= \mathrm{e}^{\mathrm{i}a_2^2 t} \cosh(a_2 x + b_2) \\ v_3 &= \mathrm{e}^{\mathrm{i}a_3^2 t} \sinh(a_3 x + b_3). \end{aligned} \tag{37}$$

Substitution of (37) into (21) gives rise to the bound state solutions. The normalization constants for each are given by

$$1\Big/\sqrt{2a_1\left(a_3^2 - a_1^2\right)\left(a_2^2 - a_1^2\right)} \tag{38a}$$

$$1\Big/\sqrt{2a_2\left(a_3^2 - a_2^2\right)\left(a_2^2 - a_1^2\right)} \tag{38b}$$

$$1\Big/\sqrt{2a_3\left(a_3^2 - a_1^2\right)\left(a_3^2 - a_2^2\right)}\,. \tag{38c}$$

The respective energy levels are given by

$$E_1 = -a_1^2 \qquad E_2 = -a_2^2 \qquad \text{and} \qquad E_3 = -a_3^2. \tag{39}$$

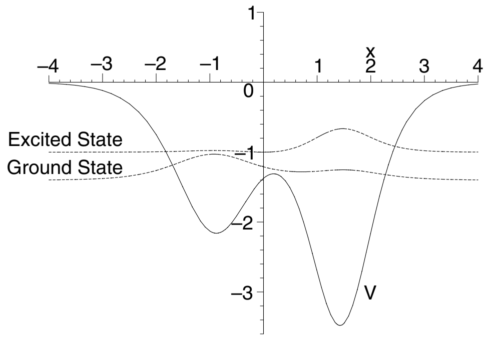


**Figure 3.** The potential with two bound states.

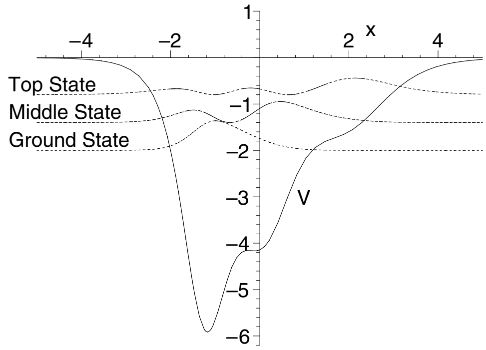


**Figure 4.** The potential with three bound states.

Figures 3 and 4 illustrate two- and three-well potentials and their appropriate probability density functions. The probability density functions are superimposed over each potential at their respective energy levels. In figure 3 the constants have been chosen as: $a_1 = 1$, $a_2 = 1.2$, $b_1 = -1$ and $b_2 = -1$; in figure 4: $a_1 = 1$, $a_2 = 1.3$, $a_3 = 1.6$, $b_1 = 0$, $b_2 = -1$ and $b_3 = 1$.

When the potential in the (1 + 1)-dimensional Schrödinger equation is given by (33), there are exactly $n$ bound states. These states can be obtained by using the $n$ generating solutions

$$\omega_k = \begin{cases} \dfrac{e^{ia_k^2 t}\sinh(a_k x + b_k)}{\sqrt{2a_k \prod_{\ell\neq k} |a_\ell^2 - a_k^2|}} & \text{if} \quad k \text{ is odd} \\ \dfrac{e^{ia_k^2 t}\cosh(a_k x + b_k)}{\sqrt{2a_k \prod_{\ell\neq k} |a_\ell^2 - a_k^2|}} & \text{if} \quad k \text{ is even.} \end{cases} \tag{40}$$

To see that the only bound state solutions for a potential obtained from the $n$ seed solutions in (29) are those obtained by using the generating solutions in (40), note that the only generating solutions, up to a constant multiple, which can be used as generating solutions and which yield a separable solution are

$$v = \begin{cases} A\,e^{i\ell^2 t}\cosh(\ell x) + B\,e^{i\ell^2 t}\sinh(\ell x) & \text{if} \quad \ell < 0 \\ A + Bx & \text{if} \quad \ell = 0 \\ A\,e^{i\ell^2 t}\cos(\ell x) + B\,e^{i\ell^2 t}\sin(\ell x) & \text{if} \quad \ell > 0. \end{cases} \tag{41}$$

Using the generating solution from (41) when $\ell = 0$ or $\ell > 0$ leads to solutions of Schrödinger's equations which are not square integrable, since asymtotically they behave, respectively, either linearly or as $c\sin(kx + d)$. Thus, the only choice remaining is the generating solutions for $\ell < 0$. If $\ell \neq a_k$, the solutions asymptotically grow exponentially, and thus are not square integrable. Therefore, it is necessary to have $\ell = a_k$ to obtain a bound state. If the generating solution coincides with a member of the seed solution family, the constructed solution will be identically zero (see Samsonov [10]). If the generating solution has the same energy but is linearly independent of the seed solutions used in constructing the potential, that is, one from (40), the resulting solution is square integrable with normalizing constant

$$n_k = \frac{1}{\sqrt{2a_k \prod_{\ell\neq k} \left|a_\ell^2 - a_k^2\right|}}. \tag{42}$$

Finally, the remaining linearly independent solutions to (24), when $V$ is given by (30), associated with the energy levels of the generating solutions given in (40) are obtained by separation of variables and using a reduction of order. These solutions grow exponentially and are again not bounded.

To justify the normalizing constants given in (42), observe that when the seed solutions are given by (29) and the generating solution is given in (41), then the solution $u_k = W_{\eta_k}/W$ is given by (21) and has the property that

$$\left(\frac{W_{\eta_k}}{W}\right)^2 = n_k^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{U_k}{W}\right). \tag{43}$$

Here $U_k$ is obtained by replacing the $k$th seed in (29) used to generate $W$ by the linearly independent $k$th solution from

$$\eta_k = \begin{cases} \mathrm{e}^{\mathrm{i}(a_k)^2 t}\sinh(a_k x + b_k) & \text{if } k \text{ is odd} \\ \mathrm{e}^{\mathrm{i}(a_k)^2 t}\cosh(a_k x + b_k) & \text{if } k \text{ is even.} \end{cases} \tag{44}$$

The seed solutions given in (29) that are used to generate the potential (30) were first cited by Matveev and Salle [8] but these results appear not to be well known in the literature. Furthermore, an extensive literature search leads us to believe that we are the first to give the normalized bound state solutions (35).

## 5. Pöschl–Teller potentials

In this section, the integral members of regular and modified Pöschl–Teller potentials are recovered by choosing the seed solutions judiciously.

### 5.1. Regular Pöschl–Teller potentials

The class of potentials commonly referred to as the Pöschl–Teller potentials is given by

$$V(x) = \frac{\ell(\ell-1)\alpha^2}{\sin^2\alpha x} + \frac{m(m-1)\alpha^2}{\cos^2\alpha x} \tag{45}$$

see, for example, Flügge [5]. By choosing the seed solutions appropriately, the integral members of regular Pöschl–Teller potentials can be recovered. For example, if the seed solution is chosen as

$$\omega_1 = \mathrm{e}^{-\mathrm{i}\alpha^2 t}\sin(\alpha x) \qquad \text{or} \qquad \omega_1 = \mathrm{e}^{-4\mathrm{i}\alpha^2 t}\sin(2\alpha x) \tag{46}$$

then the potential $V(x)$ in (4) is

$$V(x) = \frac{2\alpha^2}{\sin^2 \alpha x} \qquad \text{or} \qquad V(x) = \frac{2\alpha^2}{\sin^2 \alpha x} + \frac{2\alpha^2}{\cos^2 \alpha x} \tag{47}$$

respectively. These are the second members of the regular Pöschl–Teller potential (i.e., $\max\{l, m\} = 2$). The remaining case at this level is

$$V(x) = \frac{2\alpha^2}{\cos^2 \alpha x}. \tag{48}$$

This can be obtained from the first equation in (46) by the simple translation, $x \to x + \frac{\pi}{2\alpha}$. To obtain the higher level Pöschl–Teller potential requires using more seed solutions. If the seed solutions $\omega_j$ are chosen to be

$$\omega_j = \mathrm{e}^{-\mathrm{i}a_j^2 t} \sin(a_j x) \tag{49}$$

then there are three cases which must be distinguished. If $m = 1$ then $a_j = j\alpha$, $j = 1, 2, \ldots, \ell - 1$, if $1 < m < \ell$, then $a_j = j\alpha$, $j = 1, 2, \ldots, \ell - m$ and $a_{j+1} = a_j + 2$ for $j = \ell - m + 1, \ldots, \ell - 1$ and if $m = \ell$ then $a_j = 2j\alpha$ for $j = 1, 2, \ldots, \ell - 1$. Finally, the translation $x \to x + \frac{\pi}{2\alpha}$ gives the remaining members of the family. For example, to generate all of the modified Pöschl–Teller potential for $\max\{l, m\} = 4$, use the seed solutions

$$\omega_1 = \mathrm{e}^{-\mathrm{i}\alpha^2 t} \sin(\alpha x) \qquad \omega_2 = \mathrm{e}^{-4\mathrm{i}\alpha^2 t} \sin(2\alpha x) \qquad \omega_3 = \mathrm{e}^{-9\mathrm{i}\alpha^2 t} \sin(3\alpha x) \tag{50$a$}$$

$$\omega_1 = \mathrm{e}^{-\mathrm{i}\alpha^2 t} \sin(\alpha x) \qquad \omega_2 = \mathrm{e}^{-4\mathrm{i}\alpha^2 t} \sin(2\alpha x) \qquad \omega_3 = \mathrm{e}^{-16\mathrm{i}\alpha^2 t} \sin(4\alpha x) \tag{50$b$}$$

$$\omega_1 = \mathrm{e}^{-\mathrm{i}\alpha^2 t} \sin(\alpha x) \qquad \omega_2 = \mathrm{e}^{-9\mathrm{i}\alpha^2 t} \sin(3\alpha x) \qquad \omega_3 = \mathrm{e}^{-25\mathrm{i}\alpha^2 t} \sin(5\alpha x) \tag{50$c$}$$

$$\omega_1 = \mathrm{e}^{-4\mathrm{i}\alpha^2 t} \sin(2\alpha x) \qquad \omega_2 = \mathrm{e}^{-16\mathrm{i}\alpha^2 t} \sin(4\alpha x) \qquad \omega_3 = \mathrm{e}^{-36\mathrm{i}\alpha^2 t} \sin(6\alpha x). \tag{50$d$}$$

This gives, respectively, the Pöschl–Teller potentials

$$V(x) = \frac{12\alpha^2}{\sin^2 \alpha x} \tag{51$a$}$$

$$V(x) = \frac{12\alpha^2}{\sin^2 \alpha x} + \frac{2\alpha^2}{\cos^2 \alpha x} \tag{51$b$}$$

$$V(x) = \frac{12\alpha^2}{\sin^2 \alpha x} + \frac{6\alpha^2}{\cos^2 \alpha x} \tag{51$c$}$$

$$V(x) = \frac{12\alpha^2}{\sin^2 \alpha x} + \frac{12\alpha^2}{\cos^2 \alpha x}. \tag{51$d$}$$

The translation $x \to x + \frac{\pi}{2\alpha}$ converts (50$a$)–(50$c$) to

$$\omega_1 = \mathrm{e}^{-\mathrm{i}\alpha^2 t} \cos(\alpha x) \qquad \omega_2 = \mathrm{e}^{-4\mathrm{i}\alpha^2 t} \sin(2\alpha x) \qquad \omega_3 = \mathrm{e}^{-9\mathrm{i}\alpha^2 t} \cos(3\alpha x) \tag{52$a$}$$

$$\omega_1 = \mathrm{e}^{-\mathrm{i}\alpha^2 t} \cos(\alpha x) \qquad \omega_2 = \mathrm{e}^{-4\mathrm{i}\alpha^2 t} \sin(2\alpha x) \qquad \omega_3 = \mathrm{e}^{-16\mathrm{i}\alpha^2 t} \sin(4\alpha x) \tag{52$b$}$$

$$\omega_1 = \mathrm{e}^{-\mathrm{i}\alpha^2 t} \cos(\alpha x) \qquad \omega_2 = \mathrm{e}^{-9\mathrm{i}\alpha^2 t} \cos(3\alpha x) \qquad \omega_3 = \mathrm{e}^{-25\mathrm{i}\alpha^2 t} \cos(5\alpha x) \tag{52$c$}$$

which gives the remaining members of the family, namely

$$V(x) = \frac{12\alpha^2}{\cos^2 \alpha x} \tag{53$a$}$$

$$V(x) = \frac{12\alpha^2}{\cos^2 \alpha x} + \frac{2\alpha^2}{\sin^2 \alpha x} \tag{53b}$$

$$V(x) = \frac{12\alpha^2}{\cos^2 \alpha x} + \frac{6\alpha^2}{\sin^2 \alpha x}. \tag{53c}$$

Note that it is not necessary to convert (50*d*) as it would just interchange the two terms in (51*d*).

### *5.2. Modified Pöschl–Teller potentials*

The class of potentials commonly referred to as the modified Pöschl–Teller potentials [5] is given by

$$V(x) = \frac{\ell(\ell-1)\alpha^2}{\cosh^2 \alpha x}. \tag{54}$$

When $\ell$ is an integer, this family can be recovered by choosing the seed solutions $\omega_j$ as

$$\omega_k = \begin{cases} \mathrm{e}^{\mathrm{i}(k\alpha)^2 t} \cosh(a_k x) & \text{if } k \text{ is odd} \\ \mathrm{e}^{\mathrm{i}(k\alpha)^2 t} \sinh(a_k x) & \text{if } k \text{ is even} \end{cases} \tag{55}$$

for $k = 1, 2, \ldots, \ell$. To illustrate, if the seed solutions $\omega_1 = \mathrm{e}^{-\mathrm{i}\alpha^2 t} \cosh(\alpha x), \omega_1 = \mathrm{e}^{-\mathrm{i}4\alpha^2 t} \sinh(2\alpha x)$ and $\omega_1 = \mathrm{e}^{-\mathrm{i}9\alpha^2 t} \cosh(3\alpha x)$ are chosen, then the potential $V(x)$ in (27) is

$$V(x) = -\frac{12\alpha^2}{\cosh^2 \alpha x}. \tag{56}$$

This letter constructs a link between Schrödinger's equations with zero and nonzero potentials using an $n + 1$st-order Darboux transformation. A system of $n + 1$ nonlinear partial differential equations is constructed for the coefficients of the Darboux transformation. Via a Hopf–Cole type transformation, this system is solved. Using separable solutions of the free particle Schrödinger equation, new classes of symmetric and non-symmetric potentials are obtained. For these new potentials, bound state energy levels and corresponding bound state solutions are obtained. As special cases, the integral members of the regular and modified Pöschl–Teller potentials are recovered.

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